*Journal of Sound and Vibration* (1998) **216**(2), 215–225
*Article No*. sv981586

# ON THE SENSITIVITIES OF THE EIGENVALUES OF A VISCOUSLY DAMPED CANTILEVER CARRYING A TIP MASS

M. GÜRGÖZE

*Faculty of Mechanical Engineering, Technical University of Istanbul*, 80191 *Gümüşsuyu, Istanbul, Turkey*

(*Received* 6 *June* 1997, *and in final form* 3 *March* 1998)

This paper deals with the investigation of the sensitivity of the eigenvalues of a special mechanical system. It consists of a clamped–free Bernoulli–Euler beam carrying a tip mass. The vibrations of the beam are damped by a viscous damper which is attached to it within the span. The main concern lies in the determination of the sensitivities with respect to the changes in the magnitude of the damping constant, tip mass ratio and location of the damper attachment point, around their nominal values.



## 1. INTRODUCTION

Dynamic analysis of mechanical systems often leads to the solution of an eigenvalue problem. If the interest is in the optimization of the dynamic system one is confronted with among others the problem of effective calculation of the partial derivatives of the eigenvalues and eigenvectors with respect to some construction parameters. The calculation of these partial derivatives is referred to as "sensitivity analysis". Sensitivity analysis finds usage in various engineering applications. Some typical examples are: system identification, robust control, gradient-based optimization algorithms and approximation of the system response due to the change of a system parameter, etc. [1]. Due to the increasing importance of sensitivity analysis in engineering practice this topic is also referred to in new text books [2, 3]. In the studies [4, 5] the sensitivity of the eigenfrequencies of elastic beams with respect to small changes in the location of the in-span support is considered. In reference [6], the sensitivity of the eigenfrequencies of beams and plates with reference to changes of position of attached masses, restraining springs and spring–mass systems is discussed. In reference [7], the sensitivity of eigenvalues of a viscously damped clamped–free Bernoulli–Euler beam was investigated. The present study essentially considers the same mechanical system as in reference [7], but the system here is more general than that because a tip mass is also included. In other words, the sensitivity of eigenvalues of a clamped–free Bernoulli–Euler beam carrying a tip mass which is damped by a viscous damper will be investigated. The sensitivities to be considered are due to the changes in the magnitude of the damping constant, tip mass ratio and location of the damper attachment point.

0022–460X/98/370215 + 11 $30.00/0 

## 2. THEORY

The system to be dealt with in the present study is shown in Figure 1. It consists of essentially a cantilevered Bernoulli–Euler beam carrying a tip mass $M$. The beam is damped at the position $x = l$ by a viscous damper of damping constant $c$. Bending rigidity, length and mass per unit length of the elastic beam are $EI$, $L$ and $m$, respectively.

The partial differential equation of the free bending vibrations of a uniform beam, according to Bernoulli–Euler theory is

$$EIw^{IV}(x, t) + m\ddot{w}(x, t) + c\dot{w}(x, t)\delta(x - l) + M\ddot{w}(x, t)\delta(x - L) = 0, \qquad (1)$$

where $\delta(x)$ denotes the well known Dirac delta function. As previously stated, $c$ and $M$ are the viscous damping constant and the mass of the tip mass, respectively, and $w(x, t)$ represents the bending displacement of the beam at point $x$ and time $t$. The primes and overdots denote the partial derivatives with respect to $x$ and $t$, respectively.

An approximate series solution of equation (1) can be taken in the form

$$w(x, t) \approx \sum_{r=1}^{n} w_r(x)\eta_r(t), \qquad (2)$$

where $w_r(x)$ are the orthogonal eigenfunctions of the clamped–free beam without the tip mass and viscous damper, normalized with respect to the mass density. $\eta_r(t)$ are the unknown time dependent generalized co-ordinates.

After substitution of equation (2) into equation (1), both sides of the equation are multiplied by the $s$th eigenfunction $w_s(x)$ and integrated over the beam length $L$. By using the orthogonality property of the eigenfunctions, the following set of ordinary differential equations for the $\eta_s(t)$ is obtained

$$\ddot{\eta}_s(t) + \omega_s^2\eta_s(t) + Mw_s(L)\sum_{r=1}^{n} w_r(L)\ddot{\eta}_r(t) + cw_s(l)\sum_{r=1}^{n} w_r(l)\dot{\eta}_r(t) = 0, \qquad s = 1, \ldots, n, \qquad (3)$$

where $\omega_s$ denotes the $s$th eigenfrequency of the undamped beam without the tip mass.

If the solutions of the form

$$\eta_s(t) = \bar{\eta}_s\,e^{\lambda t}, \qquad s = 1, \ldots, n \qquad (4)$$

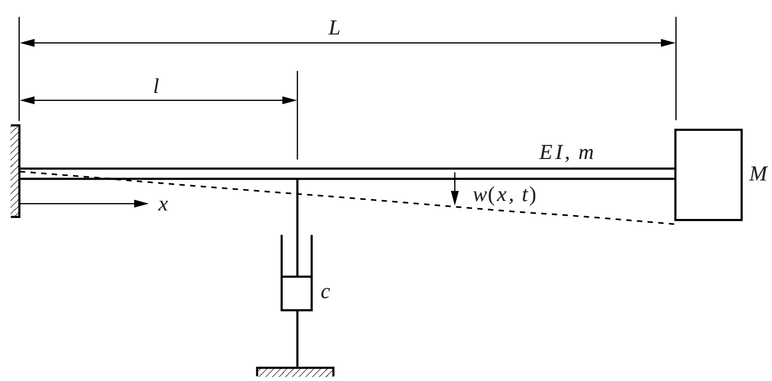


Figure 1. Viscously damped bending beam carrying a tip mass.

are substituted into the system (3) where $\lambda$ denotes an eigenvalue of the combined system in Figure 1, the following set of equations are obtained for $\bar{\eta}_s$:

$$(\lambda^2 + \omega_s^2)\bar{\eta}_s + \sum_{r=1}^{n} [M\lambda^2 w_s(L) w_r(L) + c\lambda w_s(l) w_r(l)]\bar{\eta}_r = 0, \qquad s = 1, \ldots, n. \tag{5}$$

In order to use the advantages of matrix notation one can define

$$\bar{\boldsymbol{\eta}} = [\bar{\eta}_1, \ldots, \bar{\eta}_n]^T,$$
$$\mathbf{w}(x) = [w_1(x), \ldots, w_n(x)]^T,$$
$$\boldsymbol{\Omega}^2 = \mathrm{diag}\,(\omega_i^2). \tag{6}$$

Using the definitions above, the system of equations in (5) can be written as

$$[(\lambda^2 \mathbf{I} + \boldsymbol{\Omega}^2) + M\lambda^2 \mathbf{w}(L)\mathbf{w}^T(L) + c\lambda \mathbf{w}(l)\mathbf{w}^T(l)]\bar{\boldsymbol{\eta}} = \mathbf{0}, \tag{7}$$

with $\mathbf{I}$ being the $n$-dimensional unit matrix.

The solvability condition of equations (7) yields the following characteristic equation for the mechanical system in Figure 1.

$$\det\,[(\lambda^2 \mathbf{I} + \boldsymbol{\Omega}^2) + M\lambda^2 \mathbf{w}(L)\mathbf{w}^T(L) + c\lambda \mathbf{w}(l)\mathbf{w}^T(l)] = 0. \tag{8}$$

For further investigations, it is more suitable to rewrite the characteristic equation above in terms of dimensionless quantities as

$$\det\,[\lambda^{*2}(\mathbf{I} + \beta_M \mathbf{a}(1)\mathbf{a}^T(1)) + \lambda^* \bar{c}\mathbf{a}(\bar{l})\mathbf{a}^T(\bar{l}) + \mathbf{B}] = 0, \tag{9}$$

where the following abbreviations are introduced

$$\bar{x} = x/L, \qquad \bar{l} = l/L, \qquad w_k(\bar{x}) = 1/\sqrt{mL}\, a_k(\bar{x}),$$
$$a_k(\bar{x}) = \cosh \bar{\beta}_k \bar{x} - \cos \bar{\beta}_k \bar{x} - \bar{\eta}_k(\sinh \bar{\beta}_k \bar{x} - \sin \bar{\beta}_k \bar{x}), \qquad \bar{\eta}_k = \frac{(\cosh \bar{\beta}_k + \cos \bar{\beta}_k)}{(\sinh \bar{\beta}_k + \sin \bar{\beta}_k)},$$
$$\mathbf{a}(\bar{x}) = [a_1(\bar{x}), \ldots, a_n(\bar{x})]^T, \qquad \bar{\beta}_1 = 1{\cdot}875104, \qquad \bar{\beta}_2 = 4{\cdot}694091, \ldots \text{ (see reference [8])}$$
$$\omega_k^2 = \bar{\beta}_k^4 \omega_0^2, \qquad \mathbf{B} = \mathrm{diag}\,(\bar{\beta}_k^4), \qquad \omega_0^2 = EI/mL^4, \qquad \lambda^* = \lambda/\omega_0,$$
$$\beta_M = M/mL, \qquad \bar{c} = c/mL\omega_0. \tag{10}$$

As is known from the state–space description of mechanical systems, the non-dimensional characteristic values $\lambda^*$ can also be determined as the eigenvalues of the $2n$-dimensional square matrix $\mathbf{A}^*$ defined as

$$\mathbf{A}^* = \left[\begin{array}{c:c} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{B} & -\mathbf{M}^{-1}\mathbf{D} \end{array}\right], \tag{11}$$

where the following definitions are used

$$\mathbf{M} = \mathbf{I} + \beta_M \mathbf{a}(1)\mathbf{a}^T(1), \qquad \mathbf{D} = \bar{c}\mathbf{a}(\bar{l})\mathbf{a}^T(\bar{l}). \tag{12}$$

In order to derive the sensitivity formulas, it is necessary to express the determinant in equation (9) in an analytical form. To this end one wants to use a determinantal formula from matrix theory, namely

$$\det\,(\mathbf{A} + \alpha \mathbf{d}\mathbf{d}^T) = (\det \mathbf{A})(1 + \alpha \mathbf{d}^T \mathbf{A}^{-1} \mathbf{d}), \tag{13}$$

where $\alpha$ is a scalar, $\mathbf{A}$ is a regular $n \times n$ matrix and $\mathbf{d}$ is an $n$-dimensional column vector [9]. In other words, the matrix the determinant of which is to be computed is the sum of a regular square matrix and a dyadic product multiplied by a scalar. Various forms of the formula (13) are often used in the control theory in the context of multivariable feedback and pole location [10].

By using this formula it can be shown in a straightforward manner that the following relation holds for the determinant of the sum of a regular square matrix $\mathbf{A}$ and two dyadic products multiplied by the scalars $\alpha$ and $\beta$

$$\det(\mathbf{A} + \alpha\mathbf{d}\mathbf{d}^T + \beta\mathbf{p}\mathbf{p}^T) = \det \mathbf{A}\{1 + \alpha\mathbf{d}^T\mathbf{A}^{-1}\mathbf{d} + \beta\mathbf{p}^T\mathbf{A}^{-1}\mathbf{p} + \alpha\beta(\mathbf{d}^T\mathbf{A}^{-1}\mathbf{d}\mathbf{p}^T\mathbf{A}^{-1}\mathbf{p} - \mathbf{p}^T\mathbf{A}^{-1}\mathbf{d}\mathbf{d}^T\mathbf{A}^{-1}\mathbf{p})\}. \tag{14}$$

The comparison of the expressions (9) and (14) reveals that the following correspondences hold

$$\mathbf{A} \mathrel{\hat{=}} \lambda^{*2}\mathbf{I} + \mathbf{B}, \qquad \mathbf{d} \mathrel{\hat{=}} \mathbf{a}(1), \qquad \mathbf{p} \mathrel{\hat{=}} \mathbf{a}(\bar{l}), \qquad \alpha \mathrel{\hat{=}} \beta_M\lambda^{*2}, \qquad \beta \mathrel{\hat{=}} \bar{c}\lambda^{*}. \tag{15}$$

It is obvious that the matrix $\mathbf{A}$ here is diagonal. Hence, the characteristic equation (9) can be reformulated as

$$1 + \beta_M\lambda^{*2}\sum_{k=1}^{n}\frac{a_k^2(1)}{\lambda^{*2}+\bar{\beta}_k^4} + \bar{c}\lambda^{*}\sum_{k=1}^{n}\frac{a_k^2(\bar{l})}{\lambda^{*2}+\bar{\beta}_k^4} + \beta_M\bar{c}\lambda^{*3}\left[\left(\sum_{k=1}^{n}\frac{a_k^2(1)}{\lambda^{*2}+\bar{\beta}_k^4}\right)\left(\sum_{j=1}^{n}\frac{a_j^2(\bar{l})}{\lambda^{*2}+\bar{\beta}_j^4}\right) - \left(\sum_{k=1}^{n}\frac{a_k(1)a_k(\bar{l})}{\lambda^{*2}+\bar{\beta}_k^4}\right)^2\right] = 0. \tag{16}$$

Before proceeding further, it is in order to consider the special case where the tip mass does not exist, i.e., $\beta_M = 0$.

In this case, equation (16) reduces to the simple expression

$$1 + \bar{c}\lambda^{*}\sum_{k=1}^{n}\frac{a_k^2(\bar{l})}{\lambda^{*2}+\bar{\beta}_k^4} = 0, \tag{17}$$

which leads to

$$1 + c\lambda\sum_{k=1}^{n}\frac{w_k^2(\bar{l})}{\lambda^2+\omega_k^2} = 0. \tag{18}$$

This is just the characteristic equation given in reference [7].

## 3. DERIVATION OF SENSITIVITY FORMULAS

The main contribution of the present paper is to develop closed form expressions for a particular class of viscously damped cantilever beams carrying a tip mass. Hence, the aim is not calculating the roots of the non-linear characteristic equation (16) numerically, but to provide the design engineer a set of general purpose formulae which would yield the effect of various parameters on the eigencharacteristics of the system directly. Once the related tedious manipulations are over then it is a simple matter to perform the calculations for a non-expert.

Having obtained the characteristic equation in an analytical form, one is now in a position to derive various sensitivity expressions of the eigenvalues of the system in

Figure 1. Let us begin with the sensitivity of the eigenvalues with respect to the viscous damping constant $c$.

It is easy to show that

$$\frac{\partial\lambda}{\partial c} = \frac{1}{mL}\,\lambda^{*\prime}, \tag{19}$$

where a prime denotes partial derivative with respect to the dimensionless damping constant $\bar{c}$. Differentiating the expression (16) partially with respect to $\bar{c}$ results in

$$\lambda^{*\prime} = p_1/p, \tag{20}$$

where the following abbreviations are used

$$p_1 = \frac{1}{\bar{c}}\left[1 + \beta_M\lambda^{*2}\sum_{k=1}^{n}\frac{a_k^2(1)}{\lambda^{*2}+\bar{\beta}_k^4}\right], \tag{21}$$

$$\begin{aligned} p = {} & \bar{c}\sum_{k=1}^{n}\frac{a_k^2(\bar{l})}{\lambda^{*2}+\bar{\beta}_k^4} + 2\lambda^{*}\beta_M\sum_{k=1}^{n}\frac{a_k^2(1)}{\lambda^{*2}+\bar{\beta}_k^4} + \lambda^{*2}\left[3\beta_M\bar{c}g - 2\bar{c}\sum_{k=1}^{n}\frac{a_k^2(\bar{l})}{(\lambda^{*2}+\bar{\beta}_k^4)^2}\right] \\ & -2\beta_M\lambda^{*3}\sum_{k=1}^{n}\frac{a_k^2(1)}{(\lambda^{*2}+\bar{\beta}_k^4)^2} - 2\beta_M\bar{c}\lambda^{*4}\left[\left(\sum_{k=1}^{n}\frac{a_k^2(1)}{(\lambda^{*2}+\bar{\beta}_k^4)^2}\right)\left(\sum_{j=1}^{n}\frac{a_j^2(\bar{l})}{\lambda^{*2}+\bar{\beta}_j^4}\right)\right. \\ & \left.+\left(\sum_{k=1}^{n}\frac{a_k^2(1)}{\lambda^{*2}+\bar{\beta}_k^4}\right)\left(\sum_{j=1}^{n}\frac{a_j^2(\bar{l})}{(\lambda^{*2}+\bar{\beta}_j^4)^2}\right) - 2\left(\sum_{k=1}^{n}\frac{a_k(1)a_k(\bar{l})}{\lambda^{*2}+\bar{\beta}_k^4}\right)\left(\sum_{k=1}^{n}\frac{a_k(1)a_k(\bar{l})}{(\lambda^{*2}+\bar{\beta}_k^4)^2}\right)\right], \end{aligned} \tag{22}$$

where $g$ denotes the bracket in equation (16).

Hence, it is now possible to give an approximate formula for the modified value of an eigenvalue $\lambda(c)$ if the damping constant of the damper is changed by a small amount $\Delta c$ around its nominal value $c$:

$$\lambda(c+\Delta c) \approx \lambda(c) + \left(\frac{\partial\lambda}{\partial c}\right)\Delta c = \lambda(c) + \frac{\lambda^{*\prime}}{mL}\,\Delta c. \tag{23}$$

Before going further to the derivation of other sensitivities, it is in order to consider again the special case $\beta_M = 0$, i.e., where the tip mass is not present. In this case, the partial derivative in equation (20) reduces to

$$\lambda^{*\prime} = \frac{1}{\bar{c}^2\left[\sum_{k=1}^{n}\frac{a_k^2(\bar{l})}{\lambda^{*2}+\bar{\beta}_k^4} - 2\lambda^{*2}\sum_{k=1}^{n}\frac{a_k^2(\bar{l})}{(\lambda^{*2}+\bar{\beta}_k^4)^2}\right]}. \tag{24}$$

On the other hand, from equation (17)

$$\bar{c}\sum_{k=1}^{n}\frac{a_k^2(\bar{l})}{\lambda^{*2}+\bar{\beta}_k^4} = -\frac{1}{\lambda^{*}} \tag{25}$$

is obtained. The substitution of this expression into equation (24) yields

$$\lambda^{*\prime} = -\frac{\lambda^*}{\bar{c}} \frac{1}{1 + 2\lambda^{*3}\bar{c} \sum_{k=1}^{n} \frac{a_k^2(\bar{l})}{(\lambda^{*2} + \bar{\beta}_k^4)^2}}, \tag{26}$$

which when put into equation (19) results in

$$\frac{\partial \lambda}{\partial c} = -\frac{\lambda}{c} \frac{1}{1 + 2\lambda^3 c \sum_{k=1}^{n} \frac{w_k^2(\bar{l})}{(\lambda^2 + \omega_k^2)^2}}. \tag{27}$$

This is just the sensitivity formula given in reference [7].

From a practical point of view, it can also be interesting to have the sensitivity of the eigenvalues of the system with respect to the tip mass ratio. It can be written as

$$\frac{\partial \lambda}{\partial \beta_M} = \omega_0 \frac{\partial \lambda^*}{\partial \beta_M}. \tag{28}$$

After differentiating equation (16) partially with respect to $\beta_M$

$$\frac{\partial \lambda^*}{\partial \beta_M} = \frac{p_2}{p} \tag{29}$$

is obtained where $p$ was introduced in equation (22) and $p_2$ is defined as

$$p_2 = -\lambda^{*2} \sum_{k=1}^{n} \frac{a_k^2(1)}{\lambda^{*2} + \bar{\beta}_k^4} - \bar{c}\lambda^{*3} g, \tag{30}$$

where $g$ denotes the interior of the bracket in equation (16). It is now possible to give an approximate expression for the modified value of an eigenvalue $\lambda(\beta_M)$, if the tip mass ratio is changed due to some reason by a small amount $\Delta\beta_M$ around its nominal value $\beta_M$:

$$\lambda(\beta_M + \Delta\beta_M) \approx \lambda(\beta_M) + \left(\frac{\partial \lambda}{\partial \beta_M}\right)\Delta\beta_M. \tag{31}$$

Finally, an expression can also be derived for the sensitivity of the eigenvalues with respect to the position of the damper attachment point to the beam.

In order to determine $\partial\lambda^*/\partial\bar{l}$ in

$$\frac{\partial \lambda}{\partial \bar{l}} = \omega_0 \frac{\partial \lambda^*}{\partial \bar{l}}, \tag{32}$$

one has to differentiate equation (16) partially with respect to $\bar{l}$ which results in

$$\frac{\partial \lambda^*}{\partial \bar{l}} = \frac{p_3}{p}, \tag{33}$$

where $p_3$ is defined by

$$p_3 = -c\bar{\lambda}^*\left\{\sum_{k=1}^{n} \frac{a_k(\bar{l})a_k'(\bar{l})}{\lambda^{*2}+\bar{\beta}_k^4} + \beta_M\lambda^{*2}\left[\left(\sum_{k=1}^{n}\frac{a_k^2(1)}{\lambda^{*2}+\bar{\beta}_k^4}\right)\left(\sum_{j=1}^{n}\frac{a_j(\bar{l})a_j'(\bar{l})}{\lambda^{*2}+\bar{\beta}_j^4}\right)\right.\right.$$

$$\left.\left. -\left(\sum_{k=1}^{n}\frac{a_k(1)a_k(\bar{l})}{\lambda^{*2}+\bar{\beta}_k^4}\right)\left(\sum_{k=1}^{n}\frac{a_k(1)a_k'(\bar{l})}{\lambda^{*2}+\bar{\beta}_k^4}\right)\right]\right\}, \tag{34}$$

with $a_k'(\bar{l})$ being introduced as

$$a_k'(\bar{l}) = \bar{\beta}_k[\sinh\bar{\beta}_k\bar{l} + \sin\bar{\beta}_k\bar{l} - \bar{\eta}_k(\cosh\bar{\beta}_k\bar{l} - \cos\bar{\beta}_k\bar{l})], \tag{35}$$

i.e., the derivative of $a_k(\bar{l})$ with respect to $\bar{l}$ and $p$ has to be taken again from equation (22). Hence, an approximate expression for the modified value of an eigenvalue $\lambda(\bar{l})$ if the position of the damper attachment point to the beam is changed by a small amount of $\Delta\bar{l}$ around its nominal value $\bar{l}$ is

$$\lambda(\bar{l}+\Delta\bar{l}) \approx \lambda(\bar{l}) + \left(\frac{\partial\lambda}{\partial\bar{l}}\right)\Delta\bar{l}. \tag{36}$$

## 4. NUMERICAL APPLICATIONS

This section is devoted to the numerical evaluation of the sensitivity expressions derived in the preceding section. To this end the following numerical values are chosen for the physical data of the mechanical system in Figure 1: $E = 7 \times 10^{10}$ N/m$^2$, $I = (0\cdot05 * 0\cdot005^3)/12$ m$^4$, $L = 1$ m, $mL = 0\cdot675$ kg, $\bar{l} = l/L = 0\cdot2$, $c = 5$ N/(m/s), $\beta_M = 3$. The number of the modes $n$ in expansion (2) is chosen as 10.

Table 1 gives an indication on the accuracy of the sensitivity related equation (23) in connection with equations (20)–(22). Small changes of the damping constant $c$ around its nominal value $c = 5$ N/(m/s) are taken as $\Delta c = 0\cdot5$, 1, 1·5 and 2, respectively. The complex numbers in the first columns are characteristic values $\lambda$ which are obtained as the eigenvalues of the $2n \times 2n$ matrix $\mathbf{A}^*$ defined in equation (11) and multiplied by $\omega_o$. The complex numbers in the second columns are approximate eigenvalues which are computed via the sensitivity-based formula (23) in connection with equations (20)–(22). An inspection of both columns indicates clearly that the accuracy of the formula is excellent even for larger changes of the damping constant.

Similarly, Table 2 gives an indication on the accuracy of the sensitivity-based formula (31) in connection with equations (22) and (28)–(30). Small changes of the tip mass ratio $\beta_M$ around its nominal value $\beta_M = 3$ are chosen as $\Delta\beta_M = 0\cdot001$, 0·01, 0·1 and 0·5, respectively. As in the preceding case, the first columns contain those complex numbers which are obtained as the eigenvalues of the matrix $\mathbf{A}^*$ in equation (11) multiplied by $\omega_o$. The complex numbers in the second columns are approximate eigenvalues computed via the sensitivity-based formula (31). The comparison of the complex numbers in both columns reveals clearly that equation (31) gives very accurate approximations to the eigenvalues of the modified system without having to compute the eigenvalues of the matrix $\mathbf{A}^*$ for the parameters of the modified system.

Finally, Table 3 serves to test the accuracy of the sensitivity-based formula (36) in connection with equations (22) and (32)–(35). Small changes in the location of the damper attachment point are taken as $\Delta\bar{l} = 0\cdot0005$ and 0·001. In the first columns, the eigenvalues of the matrix $\mathbf{A}^*$ in equations (11) multiplied by $\omega_o$ are collected. The complex numbers

Table 1

*Modified eigenvalues due to the change of the damping constant c by $\Delta c$*

| From equation (11) | From equation (23) |
|---|---|
| (a) $\Delta c = 0{\cdot}5$ | |
| −0·0040272 ± 7·0761439*i* | −0·0040272 ± 7·0761438*i* |
| −0·8645606 ± 115·55949*i* | −0·8645599 ± 115·55941*i* |
| −4·7401173 ± 369·87972*i* | −4·7400698 ± 369·87942*i* |
| −8·8695639 ± 769·52033*i* | −8·8694527 ± 769·52112*i* |
| −8·1726420 ± 1315·4709*i* | −8·1727081 ± 1315·4728*i* |
| −2·9717554 ± 2008·9140*i* | −2·9718020 ± 2008·9147*i* |
| −0·0007940 ± 2850·8149*i* | −0·0007940 ± 2850·8149*i* |
| −3·0596072 ± 3843·1176*i* | −3·0596082 ± 3843·1117*i* |
| −7·1857527 ± 4989·6100*i* | −7·1857547 ± 4989·6106*i* |
| −8·2254317 ± 6300·5273*i* | −8·2254754 ± 6300·5285*i* |
| (b) $\Delta c = 1$ | |
| −0·0043933 ± 7·0761450i | −0·0043933 ± 7·0761448i |
| −0·9431629 ± 115·56113i | −0·9431602 ± 115·56085i |
| −5·1714601 ± 369·88665i | −5·1712638 ± 369·88543i |
| −9·6768779 ± 769·50243i | −9·6764186 ± 769·50556i |
| −8·9150213 ± 1315·4284i | −8·9152944 ± 1315·4359i |
| −3·241503 ± 2008·8984i | −3·2416926 ± 2008·9012i |
| −0·0008662 ± 2850·8149i | −0·0008661 ± 2850·8149i |
| −3·3377441 ± 3843·1134i | −3·3377484 ± 3843·1141i |
| −7·8389854 ± 4989·5977i | −7·8389935 ± 4989·5998i |
| −8·9728089 ± 6300·5012i | −8·9729895 ± 6300·5057i |
| (c) $\Delta c = 1{\cdot}5$ | |
| −0·0047594 ± 7·0761462i | −0·0047594 ± 7·0761457i |
| −1·0217668 ± 115·56293i | −1·0217604 ± 115·56228i |
| 5·6029133 ± 369·89418i | 5·6024578 ± 369·89145i |
| −10·484450 ± 769·48296i | −10·483384 ± 769·49001i |
| −9·6572467 ± 1315·3822i | −9·6578808 ± 1315·3989i |
| −3·5111372 ± 2008·8815i | −3·5115833 ± 2008·8876i |
| −0·0009383 ± 2850·8149i | −0·0009383 ± 2850·8149i |
| −3·6158786 ± 3843·1089i | −3·6158885 ± 3843·1105i |
| −8·4922136 ± 4989·5843i | −8·4922324 ± 4989·5891i |
| −9·7200846 ± 6300·4728i | −9·7205035 ± 6300·4830i |
| (d) $\Delta c = 2$ | |
| −0·0051255 ± 7·0761475i | −0·0051255 ± 7·0761467i |
| −1·1003724 ± 115·56487i | −1·1003607 ± 115·56372i |
| −6·0344862 ± 369·90231i | −6·0336518 ± 369·89746i |
| −11·292303 ± 769·46192i | −11·290350 ± 769·47445i |
| −10·399305 ± 1315·3323i | −10·400467 ± 1315·3619i |
| −3·7806570 ± 2008·8632i | −3·7814739 ± 2008·8740i |
| −0·0010104 ± 2850·8149i | −0·0010104 ± 2850·8149i |
| −3·8940104 ± 3843·1041i | −3·8940286 ± 3843·10691i |
| −9·1454367 ± 4989·5698i | −9·1454713 ± 4989·5784i |
| −10·467250 ± 6300·4421i | −10·468018 ± 6300·4602i |

in the second columns are approximate eigenvalues obtained from the sensitivity-based formula (36). An inspection of both columns indicates clearly that the above formula yields good approximations to the eigenvalues of the modified system which is obtained when

TABLE 2

*Modified eigenvalues due to the change of the tip mass ratio $\beta_M$ by $\Delta\beta_M$*

| From equation (11) | From equation (31) |
|---|---|
| (a) $\Delta\beta_M = 0{\cdot}001$ | |
| $-0{\cdot}0036599 \pm 7{\cdot}0750501i$ | $-0{\cdot}036599 \pm 7{\cdot}0750498i$ |
| $-0{\cdot}7859542 \pm 115{\cdot}55727i$ | $-0{\cdot}7859543 \pm 115{\cdot}55727i$ |
| $-4{\cdot}3088694 \pm 369{\cdot}87263i$ | $-4{\cdot}3088694 \pm 369{\cdot}87262i$ |
| $-8{\cdot}0624875 \pm 769{\cdot}53589i$ | $-8{\cdot}0624875 \pm 769{\cdot}53589i$ |
| $-7{\cdot}4301285 \pm 1315{\cdot}5090i$ | $-7{\cdot}4301285 \pm 1315{\cdot}5090i$ |
| $-2{\cdot}7019164 \pm 2008{\cdot}9275i$ | $-2{\cdot}7019165 \pm 2008{\cdot}9275i$ |
| $-0{\cdot}0007220 \pm 2850{\cdot}8142i$ | $-0{\cdot}0007220 \pm 2850{\cdot}8142i$ |
| $-2{\cdot}7814658 \pm 3843{\cdot}1207i$ | $-2{\cdot}7814658 \pm 3843{\cdot}1206i$ |
| $-6{\cdot}5325148 \pm 4989{\cdot}6207i$ | $-6{\cdot}5325148 \pm 4989{\cdot}6207i$ |
| $-7{\cdot}4779649 \pm 6300{\cdot}5507i$ | $-7{\cdot}4779649 \pm 6300{\cdot}5507i$ |
| (b) $\Delta\beta_M = 0{\cdot}01$ | |
| $-0{\cdot}0036496 \pm 7{\cdot}0652372i$ | $-0{\cdot}0036495 \pm 7{\cdot}06521191i$ |
| $-0{\cdot}7859060 \pm 115{\cdot}55094i$ | $-0{\cdot}7859058 \pm 115{\cdot}55092i$ |
| $-4{\cdot}3088116 \pm 369{\cdot}86568i$ | $-4{\cdot}3088114 \pm 369{\cdot}86565i$ |
| $-8{\cdot}0624932 \pm 769{\cdot}52891i$ | $-8{\cdot}0624933 \pm 769{\cdot}52888i$ |
| $-7{\cdot}4301884 \pm 1315{\cdot}5021i$ | $-7{\cdot}4301887 \pm 1315{\cdot}5021i$ |
| $-2{\cdot}7019627 \pm 2008{\cdot}9208i$ | $-2{\cdot}7019629 \pm 2008{\cdot}9208i$ |
| $-0{\cdot}0007226 \pm 2850{\cdot}8077i$ | $-0{\cdot}0007226 \pm 2850{\cdot}8077i$ |
| $-2{\cdot}7814452 \pm 3843{\cdot}1146i$ | $-2{\cdot}7814450 \pm 3843{\cdot}1145i$ |
| $-6{\cdot}5325057 \pm 4989{\cdot}6153i$ | $-6{\cdot}5325057 \pm 4989{\cdot}6153i$ |
| $-7{\cdot}4779959 \pm 6300{\cdot}5464i$ | $-7{\cdot}4779960 \pm 6300{\cdot}5463i$ |
| (c) $\Delta\beta_M = 0{\cdot}1$ | |
| $-0{\cdot}0035493 \pm 6{\cdot}9693016i$ | $-0{\cdot}00354573 \pm 6{\cdot}9668327i$ |
| $-0{\cdot}7854388 \pm 115{\cdot}48953i$ | $-0{\cdot}7854218 \pm 115{\cdot}48735i$ |
| $4{\cdot}3082520 \pm 369{\cdot}79831i$ | $4{\cdot}3082314 \pm 369{\cdot}7958i$ |
| $-8{\cdot}0625491 \pm 769{\cdot}46124i$ | $-8{\cdot}0625510 \pm 769{\cdot}45877i$ |
| $-7{\cdot}4307691 \pm 1315{\cdot}4350i$ | $-7{\cdot}4307903 \pm 1315{\cdot}4325i$ |
| $-2{\cdot}7024105 \pm 2008{\cdot}8552i$ | $-2{\cdot}7024269 \pm 2008{\cdot}8528i$ |
| $-0{\cdot}0007294 \pm 2850{\cdot}7448i$ | $-0{\cdot}0007296 \pm 2850{\cdot}7425i$ |
| $-2{\cdot}7812450 \pm 3843{\cdot}0558i$ | $-2{\cdot}7812375 \pm 3843{\cdot}0536i$ |
| $-6{\cdot}5324178 \pm 4989{\cdot}5627i$ | $-6{\cdot}5324145 \pm 4989{\cdot}5608i$ |
| $-7{\cdot}4782960 \pm 6300{\cdot}5042i$ | $-7{\cdot}4783070 \pm 6300{\cdot}5026i$ |
| (d) $\Delta\beta_M = 0{\cdot}5$ | |
| $-0{\cdot}0031629 \pm 6{\cdot}5857209i$ | $-0{\cdot}0030845 \pm 6{\cdot}5295917i$ |
| $-0{\cdot}7836472 \pm 115{\cdot}25339i$ | $-0{\cdot}783270 \pm 115{\cdot}20484i$ |
| $-4{\cdot}3061097 \pm 369{\cdot}53990i$ | $-4{\cdot}3056537 \pm 369{\cdot}48570i$ |
| $-8{\cdot}0627655 \pm 769{\cdot}20199i$ | $-8{\cdot}0628074 \pm 769{\cdot}14716i$ |
| $-7{\cdot}4329938 \pm 1315{\cdot}1780i$ | $-7{\cdot}4334644 \pm 1315{\cdot}1234i$ |
| $-2{\cdot}7041251 \pm 2008{\cdot}6043i$ | $-2{\cdot}7044893 \pm 2008{\cdot}5508i$ |
| $-0{\cdot}0007555 \pm 2850{\cdot}5042i$ | $-0{\cdot}0007606 \pm 2850{\cdot}4529i$ |
| $-2{\cdot}7804790 \pm 3842{\cdot}8310i$ | $-2{\cdot}7803153 \pm 3842{\cdot}7829i$ |
| $-6{\cdot}5320814 \pm 4989{\cdot}3618i$ | $-6{\cdot}5320093 \pm 4989{\cdot}3188i$ |
| $-7{\cdot}4794439 \pm 6300{\cdot}3428i$ | $-7{\cdot}4796896 \pm 6300{\cdot}3082i$ |

the attachment point of the damper is changed slightly about its nominal position given by $\bar{l}$.

Considering the differences in the order of magnitudes of the modifications of the corresponding parameters in the three tables, it can be stated that the eigenvalues of the

TABLE 3

*Modified eigenvalues due to the change of the damper attachment point $\bar{l}$ by $\Delta\bar{l}$*

| From equation (11) | From equation (36) |
|---|---|
| (a) $\Delta\bar{l} = 0{\cdot}0005$ | |
| $-0{\cdot}0036964 \pm 7{\cdot}0761430i$ | $-0{\cdot}0036681 \pm 7{\cdot}0761429i$ |
| $-0{\cdot}7924468 \pm 115{\cdot}55805i$ | $-0{\cdot}7872538 \pm 115{\cdot}55799i$ |
| $-4{\cdot}3342582 \pm 369{\cdot}87328i$ | $-4{\cdot}3139509 \pm 369{\cdot}87337i$ |
| $-8{\cdot}0794670 \pm 769{\cdot}53503i$ | $-8{\cdot}0659157 \pm 769{\cdot}53634i$ |
| $-7{\cdot}3930999 \pm 1315{\cdot}5097i$ | $-7{\cdot}4227689 \pm 1315{\cdot}5098i$ |
| $-2{\cdot}6405926 \pm 2008{\cdot}9300i$ | $-2{\cdot}6896100 \pm 2008{\cdot}9286i$ |
| $-6{\cdot}5 \times 10^{-7} \pm 2850{\cdot}8149i$ | $-0{\cdot}000424 \pm 2850{\cdot}8149i$ |
| $-2{\cdot}8634162 \pm 3843{\cdot}1209i$ | $-2{\cdot}7978152 \pm 3843{\cdot}1212i$ |
| $-6{\cdot}5907237 \pm 4989{\cdot}6200i$ | $-6{\cdot}5443523 \pm 4989{\cdot}6210i$ |
| $-7{\cdot}3824365 \pm 6300{\cdot}5522i$ | $-7{\cdot}4591202 \pm 6300{\cdot}5514i$ |
| (b) $\Delta\bar{l} = 0{\cdot}001$ | |
| $-0{\cdot}0037321 \pm 7{\cdot}0761431i$ | $-0{\cdot}0036752 \pm 7{\cdot}07614291i$ |
| $-0{\cdot}7989665 \pm 115{\cdot}55813i$ | $-0{\cdot}7885480 \pm 115{\cdot}55801i$ |
| $-4{\cdot}3596536 \pm 369{\cdot}87314i$ | $-4{\cdot}3190260 \pm 369{\cdot}87335i$ |
| $-8{\cdot}0961182 \pm 769{\cdot}53338i$ | $-8{\cdot}0693445 \pm 769{\cdot}53601i$ |
| $-7{\cdot}3555674 \pm 1315{\cdot}5097i$ | $-7{\cdot}4154159 \pm 1315{\cdot}5098i$ |
| $-2{\cdot}5796596 \pm 2008{\cdot}9317i$ | $-2{\cdot}6773087 \pm 2008{\cdot}9290i$ |
| $-0{\cdot}0008110 \pm 2850{\cdot}8149i$ | $-0{\cdot}0001270 \pm 2850{\cdot}8149i$ |
| $-2{\cdot}9457597 \pm 3843{\cdot}1205i$ | $-2{\cdot}8141624 \pm 3843{\cdot}1211i$ |
| $-6{\cdot}6469565 \pm 4989{\cdot}6187i$ | $-6{\cdot}5561888 \pm 4989{\cdot}6208i$ |
| $-7{\cdot}2843294 \pm 6300{\cdot}5532i$ | $-7{\cdot}4402791 \pm 6300{\cdot}5516i$ |

system are much more sensitive with respect to the changes of the position of the damper than to changes of the damping constant and the tip mass ratio.

## 5. CONCLUSIONS

The present study deals with the investigation of the sensitivity of the eigenvalues of a special mechanical system consisting of a viscously damped, clamped–free Bernoulli–Euler beam carrying a tip mass. Sensitivity formulas with respect to changes in the magnitude of the damping constant, tip mass ratio and location of the damper attachment point are established. Numerical results collected in the form of various tables indicate clearly that the eigenvalues can be determined very accurately by means of sensitivity formulas obtained if the construction parameters above are changed slightly around their nominal values.

## ACKNOWLEDGMENT

The author expresses his gratitude to his friend Prof. Dr. P.C. Müller from University of Wuppertal (Germany) for his suggestions in obtaining equation (14) from equation (13).

## REFERENCES

1. Y. Z. HANG, D. T. NGUYEN and J. W. HOU 1992 *Computers & Structures* **44**, 689–692. An alternative formulation for design sensitivity analysis of linear structural dynamic systems.

2. D. J. INMAN 1989 *Vibration With Control Measurement and Stability*. Englewood Cliffs, NJ: Prentice-Hall.
3. R. T. HAFTKA and Z. GÜRDAL 1992 *Elements of Structural Optimization*. Dordrecht: Kluwer; third revised and expanded edition.
4. J. W. HOU and C. H. CHUANG 1990 16*th Automation Conference*, *ASME Transactions Chicago*, *IL*. Design sensitivity analysis and optimization of vibrating beams with variable support locations.
5. M. GÜRGÖZE, K. ÖZGÜR and H. EROL 1995 *Computers & Structures* **56**, 85–92. On the eigenfrequencies of a cantilevered beam with a tip mass and in-span support.
6. B. P. WANG 1993 *American Institution of Aeronautics and Astronautics Journal* **31**, 791–794. Eigenvalue sensitivity with respect to location of internal stiffness and mass attachments.
7. M. GÜRGÖZE and A. ÖZER 1997 *Zeitschrift für Angewandte Mathematik und Mechanik* **77**, 235–237. Sensitivitaet der Eigenwerte eines gedaempften Euler–Bernoulli Balkens in Bezug auf Daempfergrössen.
8. L. A. PARNELL and M. H. COBBLE 1976 *Journal of Sound and Vibration* **44**, 499–511. Lateral displacements of a vibrating cantilever beam with a concentrated mass.
9. F. R. GANTMACHER 1979 *The Theory of Matrices*, Vols 1 and 2. New York: Chelsea Publ.
10. M. GÜRGÖZE 1996 *International Journal of Mathematical Education in Science & Technology* **27**, 456–459. Proof of a formula on the evaluation of the determinant of a special matrix by the mathematical induction.